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MAGNETOOPTICAL, ELECTROOPTICAL AND PHOTOELASTIC EFFECTS IN AN ELASTIC POLARIZABLE AND MAGNETIZABLE ISOTROPIC CONTINUUM

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Abstract—The magnetooptical, electrooptical and photoelastic behaviour of an elastic polarizable and magnetizable isotropic continuum are investigated from a dynamical point of view, starting from balance equations and constitutive relations. The most original result of the theory is the fact that the continuum exhibits the Cotton–Mouton effect, together with linear birefringence of transverse sound waves. This is compared with experimental data and quantum theory results.

As expected, the continuum does not exhibit Faraday rotation.

1. INTRODUCTION

ELECTROOPTICAL and magnetooptical effects have played, in the beginning of this century, an essential part in the development of the electromagnetic theory of light.

At present, the investigation of these effects undergoes a renewal in relationship with fundamental research in solid state physics or technological development such as electromagnetooptical devices in laser techniques or electromagnetooptical reading of computer memories.

No satisfactory theory for this effect has existed till now in the frame of the mechanics of continuous media.

Despite an attempt of Toupin [1] to include some of these magnetooptical effects through the completely arbitrary introduction, in the constitutive relations, of a gyration vector, and an attempt of kinematical nature of Rivlin and Carroll [2, 3], no rational derivation of these effects from the general balance principles and constitutive equations of the mechanics of continuous media, could be found in the literature.

This can be achieved, as shown in this paper, by use of an adequate energy–momentum tensor, as derived by Mayne and Boulanger [4], with proper constitutive relations.

This can also be obtained, with a very different point of view, by construction of a microscopic Lagrangian followed by a passage to continuum limit. This has been done by Lax and Nelson [34] in the nonmagnetic case, who give account for photoelastic, electrooptic, and various other coupling effects.

The propagation mode of electromagnetic waves in a medium is altered by the action of an external applied electric or magnetic field, or by the existence of a strain field. The latter effect, called Brewster's dynamooptical effect, constitutes the basis for the photoelastic analysis of structures.

When propagating in a direction perpendicular to the external applied magnetic field, light undergoes a linear birefringence, proportional by first approximation, to the square of the field (l being the length of the path)

$$n_{\parallel} - n_{\perp} = C_M l B^2 \tag{1}$$

where C_M is referred as the Cotton–Mouton constant. This effect is called Voigt–Cotton–Mouton effect, after its discovery by Voigt in gases, by Cotton–Mouton in liquids and solids. The microscopic origin of the effect is different in the two cases.

When propagating along the magnetic field direction, light undergoes a circular birefringence, i.e. there is a different velocity for right and left circularly polarized light, giving rise to a rotation ψ of the polarization plane, proportional to the strength of the field. (Faraday effect)

$$n_R - n_L/\psi = C_V l B \tag{2}$$

where C_V is called the Verdet constant.

In a similar way, linear birefringence occurs for a light beam travelling perpendicular to an external electrical field, and is called the Kerr effect

$$n_{\parallel} - n_{\perp} = C_K l E^2 \tag{3}$$

while the corresponding effect for propagation along the electrical field direction, sometimes called the Pockels effect, has been observed only in anisotropic bodies.

The magnetooptical Kerr effects are related to the changes in reflectance properties of a boundary of the medium, due to an external magnetic field, which may be parallel or perpendicular to the reflecting surface and to the plane of incidence. This effect will not be investigated in this paper.

2. BASIC THEORY OF RHEOOPTICAL INTERACTION IN AN ELASTIC POLARIZABLE AND MAGNETIZABLE CONTINUUM

The interaction of a nonconducting, polarizable and magnetizable continuum with an electromagnetic field can be described on the ground of an expression for the total momentum–energy tensor [4]. This gives one, at the classical approximation, the balance equations [5]:

Balance of momentum

$$\rho \frac{\mathrm{d}\bar{\mathbf{v}}}{\mathrm{d}t} = \operatorname{div} \bar{\bar{\sigma}} - \bar{\mathbf{E}}'(\operatorname{div} \bar{\mathbf{P}}') + \frac{1}{c} \underset{v}{\mathscr{L}} \bar{\mathbf{P}}' \times \bar{\mathbf{B}}' - \frac{1}{c} \underset{v}{\mathscr{L}} \bar{\mathbf{P}}' \times \bar{\mathbf{M}}' + \frac{1}{c} \bar{\mathbf{E}}' \times \underset{v}{\mathscr{L}} \bar{\mathbf{M}}'$$
$$- \operatorname{grad}(\bar{\mathbf{B}}'\bar{\mathbf{M}}') - \operatorname{rot} \bar{\mathbf{M}}' \times \bar{\mathbf{M}}' - \bar{\mathbf{B}}' \operatorname{div} \bar{\mathbf{M}}' + \frac{1}{c}(\operatorname{div} \bar{\mathbf{v}})(\bar{\mathbf{E}}' \times \bar{\mathbf{M}}') \tag{4}$$

ρ: mass density, $\bar{\mathbf{v}}$: velocity, $\bar{\bar{\boldsymbol{\sigma}}}$: stress tensor,

$$\bar{\mathbf{E}}' = \bar{\mathbf{E}} + \frac{1}{c}\bar{\mathbf{v}} \times \bar{\mathbf{B}} \qquad \bar{\mathbf{P}}' = \bar{\mathbf{P}} - \frac{1}{c}\bar{\mathbf{v}} \times \bar{\mathbf{M}}$$

$$\bar{\mathbf{B}}' = \bar{\mathbf{B}} - \frac{1}{c}\bar{\mathbf{v}} \times \bar{\mathbf{E}} \qquad \bar{\mathbf{M}}' = \bar{\mathbf{M}} + \frac{1}{c}\bar{\mathbf{v}} \times \bar{\mathbf{P}}$$

where $\bar{\mathbf{E}}$ = electric field, $\bar{\mathbf{B}}$ = magnetic flux density, $\bar{\mathbf{P}}$ = polarization (polar density), $\bar{\mathbf{M}}$ = magnetization (axial vector). The latter fields are expressed in the inertial frame, the primed quantities being expressed in the comoving frame. The $\mathscr{L}$ie derivatives of $\bar{\mathbf{P}}'$ and $\bar{\mathbf{M}}'$ are given by

$$\underset{v}{\mathscr{L}}\,\bar{\mathbf{P}}' = \partial_t \bar{\mathbf{P}}' + \mathrm{rot}(\bar{\mathbf{P}}' \times \bar{\mathbf{v}}) + (\mathrm{div}\,\bar{\mathbf{P}}')\bar{\mathbf{v}}$$

$$\underset{v}{\mathscr{L}}\,\bar{\mathbf{M}}' = \partial_t \bar{\mathbf{M}}' + (\mathrm{div}\,\bar{\mathbf{M}}')\bar{\mathbf{v}} - \bar{\mathbf{M}}'(\mathrm{div}\,\bar{\mathbf{v}}) + \mathrm{rot}(\bar{\mathbf{M}}' \times \bar{\mathbf{v}})$$

Balance of angular momentum

$$\bar{\bar{\boldsymbol{\sigma}}} = \bar{\bar{\boldsymbol{\sigma}}}^T \tag{5}$$

Balance of energy

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \sigma^{\mu}_{\lambda} v^{\lambda}_{,\mu} + \psi^{\lambda}_{,\lambda} + \bar{\mathbf{E}}'\,\underset{v}{\mathscr{L}}\,\bar{\mathbf{P}}' + \bar{\mathbf{B}}'\,\underset{v}{\mathscr{L}}\,\bar{\mathbf{M}}' \tag{6}$$

where ε = internal energy per unit mass
ψ^{λ} = heat flux.

Elastic dielectric media. In this paper, we consider only elastic transparent dielectrics in isothermal evolution, where ε, $\bar{\bar{\boldsymbol{\sigma}}}$, $\bar{\mathbf{E}}'$, $\bar{\mathbf{B}}'$ depend only on $\bar{\mathbf{P}}'$, $\bar{\mathbf{M}}'$ and deformation gradients $\partial x^k/\partial X^A \equiv x^k_{,A}$.

For vanishing magnetization [$\bar{\mathbf{M}}' = 0$], the theory reduces to Toupin's theory for polarizable elastic dielectrics with zero gyration vector ($\bar{\mathbf{G}} = 0$) [1].

We define the vectors Π' and $\mathscr{M}'$ of components

$$\Pi'^A = |(x/X)| X^A_{,k} P'^k \tag{7}$$

$$\mathscr{M}'^A = X^A_{,k} M'^k \tag{8}$$

with

$$|(x/X)| \equiv \det|x^i_{,A}|.$$

The principle of material frame-indifference requires that ε depends on $x^k_{,A}$ through the components of Green's strain tensor

$$\varepsilon = \tilde{\varepsilon}(E_{AB}, \Pi'^A \mathscr{M}'^A) \tag{9}$$

where

$$E_{AB} = \tfrac{1}{2}(g_{\lambda\mu} x^{\lambda}_{,A} x^{\mu}_{,B} - G_{AB}).$$

Using the identities

$$\dot{\Pi}^A = |(x/X)| X^A_{,k} \underset{v}{\mathscr{L}} P'^k$$

$$\dot{\mathscr{M}}'^A = X^A_{,k} \underset{v}{\mathscr{L}} M'^k,$$

the energy balance can be transformed into

$$\rho\frac{\partial\tilde{\varepsilon}}{\partial E_{AB}} x_{k,A}\dot{x}^k_{,B} + \rho\frac{\partial\tilde{\varepsilon}}{\partial \Pi'^A}\dot{\Pi}'^A + \rho\frac{\partial\tilde{\varepsilon}}{\partial\mathscr{M}'^A}\dot{\mathscr{M}}'^A = \sigma^l_k\dot{x}^k_{,A}X^A_{,l} + E'_k x^k_{,A}|(x/X)|^{-1}\dot{\Pi}'^A + B'_k x^k_{,A}\dot{\mathscr{M}}'^A$$

($\bar{\psi} = 0$, the system being nondissipative and in isothermal evolution). Identifying the coefficients of $\dot{x}^k_{,A}$, $\dot{\Pi}'^A$ and $\dot{\mathscr{M}}'^A$, together with

$$\rho_0 = \rho|(x/X)| \text{ and } (5)$$

$$\sigma^{kl} = \sigma^{lk} = \rho\frac{\partial\tilde{\varepsilon}}{\partial E_{AB}} x^k_{,A}x^l_{,B} \tag{10}$$

$$E'_k = \rho_0\frac{\partial\tilde{\varepsilon}}{\partial\Pi'^A}X^A_{,k} \tag{11}$$

$$B'_k = \rho\frac{\partial\tilde{\varepsilon}}{\partial\mathscr{M}'^A}X^A_{,k} \tag{12}$$

Isotropic dielectrics. For isotropic dielectric media, the function $\tilde{\varepsilon}$ is an invariant under the full orthogonal group of the strain tensor E_{AB}, of vector density Π'^A and axial vector $\mathscr{M}'^A$.

Considering the skew-symmetrical tensor associated with the axial vector $\mathscr{M}'^A$, one needs an $E3$-representation of an invariant scalar depending on a second order symmetric tensor, a second order skew-symmetric tensor and a vector. Wang [6] has derived, for the full orthogonal group, a complete and irreducible set of such invariants. In this special case, $\tilde{\varepsilon}$ depends on the 14 invariants

$$\begin{array}{lll}
I_1 = \operatorname{tr} \mathrm{E} & I_2 = \operatorname{tr} E^2 & I_3 = \operatorname{tr} E^3 \\
I_4 = \Pi'^2 & I_5 = \mathscr{M}'^2 & I_6 = \Pi' . \bar{\mathbf{E}}\Pi' \\
I_7 = \Pi' . \mathbf{E}^2\Pi' & I_8 = \bar{\mathscr{M}}' . \bar{\mathbf{E}}\bar{\mathscr{M}}' & I_9 = \bar{\mathscr{M}}' . \mathbf{E}^2\bar{\mathscr{M}}' \\
I_{10} = \bar{\mathbf{E}}\bar{\mathscr{M}}'[\bar{\mathbf{E}}^2\bar{\mathscr{M}}' \times \bar{\mathscr{M}}'] & I_{11} (\Pi' . \bar{\mathscr{M}}')^2 & I_{12} = \bar{\mathbf{E}}\Pi' . (\bar{\mathscr{M}}' \times \Pi') \\
I_{13} = \bar{\mathbf{E}}^2\Pi' . (\bar{\mathscr{M}}' \times \Pi') & I_{14} = (\Pi' . \bar{\mathscr{M}}')[\bar{\mathbf{E}}\bar{\mathscr{M}}' . (\bar{\mathscr{M}}' \times \Pi')]. &
\end{array} \tag{13}$$

In this paper, we consider only isotropic media.

Perturbation of an equilibrium state by weak fields and small deformations. In order to obtain the rheooptical effects in such a medium, we follow the method outlined by Toupin [1], linearizing all equations about an initial equilibrium time-independent solution

$${}_0x^\lambda(X) \qquad {}_0\Pi^A \qquad {}_0\mathscr{M}^A \qquad {}_0E_\lambda \qquad {}_0B_\lambda$$

(quantities with preceding zero must be calculated in this state). To simplify, we consider only homogeneous equilibrium states where Π, $\bar{\mathscr{M}}$, $\bar{\mathbf{E}}$, $\bar{\mathbf{B}}$ are constants, and ${}_0x^\lambda(X)$ linear.

Let

$$\begin{aligned} x^\lambda(X,t) &= {}_0x^\lambda + \delta_X x^\lambda(X,t) \\ \Pi'^A(X,t) &= {}_0\Pi^A + \delta_X \Pi^A(X,t) \\ \mathcal{M}'^A(X,t) &= {}_0\mathcal{M}^A + \delta_X \mathcal{M}'^A(X,t) \\ E'_\lambda(x,t) &= {}_0E_\lambda + \delta_x E'_\lambda(x,t) \\ B'_\lambda(x,t) &= {}_0B_\lambda + \delta_x B'_\lambda(x,t) \end{aligned} \tag{14}$$

be a time dependent solution.

δ_X and δ_x are respectively the variations at constant X (Lagrangian variation) or at constant x (Eulerian variation). We now calculate the variation equations, linear in the small displacements and weak fields.

Equation of motion

Multiplication of (4) by $|(x/X)|$, and computation of its Lagrangian variation, yields, using the identity

$$[x^\mu_{,A}|(x/X)|^{-1}]_{,\mu} = 0$$

$$\begin{aligned} {}_0\rho\ddot{u}_\lambda = {} & Z^\mu_{\lambda,\mu} - {}_0E_\lambda \operatorname{div} \bar{\mathbf{p}}' + \frac{1}{c}(\dot{\bar{\mathbf{p}}}' \times {}_0\bar{\mathbf{B}})_\lambda - \frac{1}{c}(\dot{\bar{\mathbf{p}}}' \times {}_0\bar{\mathbf{M}})_\lambda \\ & - {}_0M^\mu(m'_{\lambda,\mu} - m'_{\mu,\lambda} + u_{\lambda,\alpha\mu}\, {}_0M^\alpha - u^\mu_{,\alpha\lambda}\, {}_0M^\alpha) \\ & - b'_{\mu,\lambda}\, {}_0M^\mu - {}_0B_\mu(u^\mu_{,\alpha\lambda}\, {}_0M^\alpha + m'^\mu_{,\lambda}) \\ & + \frac{1}{c}({}_0\bar{\mathbf{E}} \times \dot{\bar{\mathbf{m}}}')_\lambda + \frac{1}{c}(\operatorname{div} \dot{\bar{\mathbf{u}}})({}_0\bar{\mathbf{E}} \times {}_0\bar{\mathbf{M}})_\lambda - {}_0B_\lambda(u^\mu_{,\alpha\mu}\, {}_0M^\alpha + m'^\mu_{,\mu}) \end{aligned} \tag{15}$$

where

$$Z^{\lambda\mu} \equiv |{}_0(x/X)|^{-1}\, {}_0x^\mu_{,A}\delta_X T^\lambda_A \tag{16}$$

$$m'^\lambda = {}_0x^\lambda_{,A}\delta_X \mathcal{M}'^A \tag{17}$$

$$p'^\lambda = {}_0|(x/X)|^{-1}\, {}_0x^\lambda_{,A}\delta_X \Pi'^A \tag{18}$$

$$\delta_X x^\lambda = u^\lambda$$

$$\delta_x E'_\lambda = e'_\lambda$$

$$\delta_x B'_\lambda = b'_\lambda$$

$$T^{\lambda A} = |(x/X)|\sigma^{\lambda\mu}X^A_{,\mu} = \rho_0 \frac{\partial \tilde{\varepsilon}}{\partial E_{AB}} x^\lambda_{,B} \tag{19}$$

(Piola–Kirchhoff stress tensor).

$$\delta_X E'_\lambda = \delta_x E'_\lambda$$

$$\delta_X|(x/X)| = \frac{\rho_0}{{}_0\rho} u^\mu_{,\mu}$$

$$\delta_X(M'_{\lambda,\mu}) = m'^\lambda_{,\mu} + u^\lambda_{,\nu\mu}\, {}_0M^\nu$$

$$\rho_0 = {}_0\rho\, {}_0|(x/X)|.$$

Mechanical constitutive relations

From (19)

$$(16) \rightarrow Z^{\lambda\mu} = {}_0\sigma^{\mu\nu}u^{\lambda}_{,\nu} - {}_0C^{\lambda\mu\alpha\nu}u_{\alpha,\nu} - {}_0S^{\lambda\mu}_{\nu}p'^{\nu} - {}_0X^{\lambda\mu}_{\nu}m'^{\nu} \tag{20}$$

with

$${}_0C^{\lambda\mu\nu\alpha} = \rho_0\, {}_0\left(\frac{\partial^2\tilde{\varepsilon}}{\partial E_{AB}\partial E_{CD}}\right) {}_0x^{\mu}_{,B}\, {}_0x^{\lambda}_{,A}\, {}_0x^{\alpha}_{,C}\, {}_0x^{\nu}_{,D} \tag{21}$$

$${}_0S^{\lambda\mu}_{\nu} = \rho_0\, {}_0\left(\frac{\partial^2\tilde{\varepsilon}}{\partial \Pi'^{A}\partial E_{CD}}\right) {}_0x^{\lambda}_{,C}\, {}_0x^{\mu}_{,D}\, {}_0X^{A}_{,\nu} \tag{22}$$

$${}_0X^{\lambda\mu}_{\nu} = {}_0\rho\, {}_0\left(\frac{\partial^2\tilde{\varepsilon}}{\partial E_{CD}\partial \mathcal{M}'^{A}}\right) {}_0x^{\lambda}_{,C}\, {}_0x^{\mu}_{,D}\, {}_0X^{A}_{,\nu}\,. \tag{23}$$

Maxwell's equation

Computation of $\delta_x\bar{\mathbf{E}}'$, $\delta_x\bar{\mathbf{B}}'$, $\delta_X\bar{\mathbf{P}}'$, $\delta_X\bar{\mathbf{M}}'$ yields

$$\begin{aligned} \bar{\mathbf{e}}' &= \bar{\mathbf{e}} + \frac{1}{c}\dot{\mathbf{u}} \times {}_0\bar{\mathbf{B}} \qquad & \bar{\mathbf{b}}' &= \bar{\mathbf{b}} - \frac{1}{c}\dot{\mathbf{u}} \times {}_0\bar{\mathbf{E}} \\ \bar{\mathbf{p}}' &= \bar{\mathbf{p}} - \frac{1}{c}\dot{\mathbf{u}} \times {}_0\bar{\mathbf{M}} \qquad & \bar{\mathbf{m}}' &= \bar{\mathbf{m}} + \frac{1}{c}\dot{\mathbf{u}} \times {}_0\bar{\mathbf{P}}. \end{aligned} \tag{24}$$

For a polarizable and magnetizable dielectric, Maxwell's equations are

$$\operatorname{div} \bar{\mathbf{B}} = 0 \tag{25}$$

$$\operatorname{div}(\bar{\mathbf{E}} + \bar{\mathbf{P}}) = 0 \tag{26}$$

$$\operatorname{rot} \bar{\mathbf{E}} + \frac{1}{c}\partial_t \bar{\mathbf{B}} = 0 \tag{27}$$

$$\operatorname{rot}(\bar{\mathbf{B}} - \bar{\mathbf{M}}) - \frac{1}{c}\partial_t(\bar{\mathbf{E}} + \bar{\mathbf{P}}) = 0. \tag{28}$$

Eulerian variations of these equations are, respectively

$$\operatorname{div} \bar{\mathbf{b}} = 0 \tag{29}$$

$$\operatorname{div} \bar{\mathbf{e}} = -\operatorname{div} \bar{\mathbf{p}} - ({}_0P^{\mu}u^{\lambda}_{,\mu} - {}_0P^{\lambda}u^{\mu}_{,\mu})_{,\lambda} \tag{30}$$

since

$$\delta_x P^{\lambda} = \delta_X P^{\lambda} - {}_0P^{\lambda}_{,\mu}\delta_X x^{\mu} = \delta_X P^{\lambda}$$

$$P^{\lambda} = |(x/X)|^{-1} x^{\lambda}_{,A}\, \Pi^{A}$$

$$\delta_x P^{\lambda} = p^{\lambda} + {}_0P^{\mu}u^{\lambda}_{,\mu} - {}_0P^{\lambda}u^{\mu}_{,\mu}$$

$$\operatorname{rot} \bar{\mathbf{e}} + \frac{1}{c}\dot{\bar{\mathbf{b}}} = 0 \tag{31}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta_x\overline{\mathbf{B}}) = \frac{\partial}{\partial t}(\delta_x\overline{\mathbf{B}})$$

$$\text{rot}\,\overline{\mathbf{b}}-\frac{1}{c}\dot{\mathbf{e}} = \text{rot}\,\overline{\mathbf{m}}+\text{rot}({}_0M^\mu\overline{\mathbf{u}}_{,\mu})+\frac{1}{c}\dot{\overline{\mathbf{p}}}+\frac{1}{c}\,{}_0P^\mu\dot{\overline{\mathbf{u}}}_{,\mu}-\frac{1}{c}\,{}_0\overline{\mathbf{P}}\,\text{div}\,\dot{\mathbf{u}} \tag{32}$$

since

$$\begin{aligned}\delta_x M^\lambda &= \delta_X M^\lambda\\ M^\lambda &= x^\lambda_{,A}\mathcal{M}^A\\ m^\lambda &= {}_0x^\lambda_{,A}\delta_X\mathcal{M}^A\\ \delta_x M^\lambda &= m^\lambda+{}_0M^\mu u^\lambda_{,\mu}.\end{aligned}$$

Electromagnetic constitutive relations

Using (24), the eulerian variation of the electromagnetic equations (11), (12) are

$$e_\lambda+\frac{1}{c}(\dot{\mathbf{u}}\times{}_0\overline{\mathbf{B}})_\lambda = {}_0S^{\mu\nu}_\lambda u_{\mu,\nu}+{}_0T_{\lambda\mu}p'^\mu+{}_0W_{\lambda\mu}m'^\mu-{}_0E_\mu u^\mu_{,\lambda} \tag{33}$$

where

$${}_0T_{\lambda\mu} = \frac{\rho_0^2}{{}_0\rho}\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial\Pi'^A\partial\Pi'^B}\right){}_0X^A_{,\lambda}\,{}_0X^B_{,\mu} \tag{34}$$

$${}_0W_{\lambda\mu} = {}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial\mathcal{M}'^B\partial\Pi'^A}\right){}_0X^A_{,\lambda}\,{}_0X^B_{,\mu} \tag{35}$$

$$b_\lambda-\frac{1}{c}(\dot{\mathbf{u}}\times{}_0\overline{\mathbf{E}})_\lambda = {}_0X^{\mu\nu}_\lambda u_{\mu,\nu}+{}_0W_{\mu\lambda}p'^\mu+{}_0Y_{\lambda\mu}m'^\mu-{}_0B_\mu u^\mu_{,\lambda}-{}_0B_\lambda u^\mu_{,\mu} \tag{36}$$

with

$${}_0Y_{\lambda\mu} = {}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial\mathcal{M}'^B\partial\mathcal{M}'^A}\right){}_0X^A_{,\lambda}\,{}_0X^B_{,\mu}. \tag{37}$$

3. THE PHOTOELASTIC EFFECT

We consider an initial equilibrium state

$${}_0x^\lambda = {}_0x^\lambda(X) \qquad {}_0\overline{\mathbf{E}} = {}_0\overline{\mathbf{P}} = {}_0\overline{\mathbf{M}} = {}_0\overline{\mathbf{B}} = 0.$$

For an isotropic dielectric, the function $\tilde{\varepsilon}$ depends on the 14 invariants (13) so that

$${}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial\Pi'^A}\right) = {}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial\mathcal{M}'^A}\right) = 0.$$

The constitutive relations (11), (12) and Maxwell's equations are trivially satisfied. The equation of motion and the constitutive relation (10) reduce to the classical equations of the nonlinear elasticity, without electromagnetic interaction.

On the other hand

$$ {}_0S^{\mu\nu}_{\lambda} = {}_0X^{\mu\nu}_{\lambda} = {}_0W_{\lambda\mu} = 0 \tag{38} $$

$$ {}_0T_{\lambda\mu} = 2\frac{\rho_0^2}{{}_0\rho}\left(\frac{\partial\tilde{\varepsilon}}{\partial I_4}\right)_0 {}_0C_{\lambda\mu} + 2\frac{\rho_0^2}{{}_0\rho}\left(\frac{\partial\tilde{\varepsilon}}{\partial I_6}\right)_0 {}_0e_{\lambda\mu} + 2\frac{\rho_0^2}{{}_0\rho}\left(\frac{\partial\tilde{\varepsilon}}{\partial I_7}\right)_0 {}_0e^2_{\lambda\mu} \tag{39} $$

$$ {}_0Y_{\lambda\mu} = 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right)_0 {}_0C_{\lambda\mu} + 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_8}\right)_0 {}_0e_{\lambda\mu} + 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_9}\right)_0 {}_0e^2_{\lambda\mu} \tag{40} $$

$$ {}_0C_{\lambda\mu} = G_{AB}\,{}_0X^A_{,\lambda}\,{}_0X^B_{,\mu} $$

$$ {}_0e_{\lambda\mu} = \tfrac{1}{2}(g_{\lambda\mu} - {}_0C_{\lambda\mu}). $$

The equations for $\bar{\mathbf{e}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{m}}$ (29) to (32) separate from the equation (15) for $\bar{\mathbf{u}}$. This fact implies that, to the first order, weak fields do not give rise to any displacement, and the photoelastic effect can therefore be investigated from the Maxwell and electromagnetic constitutive equations (29) to (34) alone

$$ \operatorname{div}\bar{\mathbf{b}} = 0 \tag{41} $$

$$ \operatorname{rot}\bar{\mathbf{e}} + \frac{1}{c}\dot{\bar{\mathbf{b}}} = 0 \tag{42} $$

$$ \operatorname{div}\bar{\mathbf{e}} = -\operatorname{div}\bar{\mathbf{p}} \tag{43} $$

$$ \operatorname{rot}\bar{\mathbf{b}} - \frac{1}{c}\dot{\bar{\mathbf{e}}} = \operatorname{rot}\bar{\mathbf{m}} + \frac{1}{c}\dot{\bar{\mathbf{p}}} \tag{44} $$

$$ e_\lambda = {}_0T_{\lambda\mu}p^\mu \tag{45} $$

$$ b_\lambda = {}_0Y_{\lambda\mu}m^\mu. \tag{46} $$

One seeks "plane wave solutions" of this partial derivative system

$$ \bar{\mathbf{a}} = \operatorname{Re}\tilde{\mathbf{a}}\,e^{i\omega[t-(n/c)\bar{\mathbf{s}}\bar{\mathbf{r}}]} \tag{47} $$

where $\tilde{\mathbf{a}}$ is the complex vector amplitude, $\bar{\mathbf{s}}$ the unit normal vector, $\omega/2\pi$ the frequency and n the complex refractive index.

Substitution yields the algebraic system

$$ n\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}} - \tilde{\mathbf{b}} = 0 \tag{48} $$

$$ n\bar{\bar{\mathbf{S}}}\tilde{\mathbf{b}} + \tilde{\mathbf{e}} = -\tilde{\mathbf{p}} + n\bar{\bar{\mathbf{S}}}\tilde{\mathbf{m}} \tag{49} $$

$$ \tilde{\mathbf{e}} = {}_0\bar{\bar{\mathbf{T}}}\tilde{\mathbf{p}} \tag{50} $$

$$ \tilde{\mathbf{b}} = {}_0\bar{\bar{\mathbf{Y}}}\tilde{\mathbf{m}} \tag{51} $$

where $\bar{\bar{\mathbf{S}}}$ is the skew-symmetric tensor associated to vector $\bar{\mathbf{s}}$ by

$$ \bar{\bar{\mathbf{S}}}\tilde{\mathbf{a}} = \bar{\mathbf{s}}\times\tilde{\mathbf{a}}. $$

If ${}_0\bar{\bar{\mathbf{T}}}$ and ${}_0\bar{\bar{\mathbf{Y}}}$ are nonsingular, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{m}}$, $\tilde{\mathbf{b}}$ can be eliminated from the system (48), (51)

$$[n^2\bar{\bar{\mathbf{S}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}+(\bar{\bar{\mathbf{I}}}+{}_0\bar{\bar{\mathbf{T}}}^{-1})]\tilde{\mathbf{e}} = 0. \tag{52}$$

The refractive indices will be solutions of the algebraic equation

$$\det(\bar{\bar{\mathbf{A}}}+n^2\bar{\bar{\mathbf{B}}}) = 0 \quad \text{with} \quad \bar{\bar{\mathbf{A}}} = \bar{\bar{\mathbf{I}}}+{}_0\bar{\bar{\mathbf{T}}}^{-1}$$

and

$$\bar{\bar{\mathbf{B}}} = \bar{\bar{\mathbf{S}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}.$$

Since $|\det B| = 0$, this equation reads

$$[\tfrac{1}{2}\operatorname{tr}\bar{\bar{\mathbf{A}}}(\operatorname{tr}^2\bar{\bar{\mathbf{B}}}-\operatorname{tr}\bar{\bar{\mathbf{B}}}^2)+\operatorname{tr}\bar{\bar{\mathbf{A}}}\bar{\bar{\mathbf{B}}}^2-\operatorname{tr}\bar{\bar{\mathbf{B}}}\operatorname{tr}\bar{\bar{\mathbf{A}}}\bar{\bar{\mathbf{B}}}]n^4$$
$$+[\tfrac{1}{2}\operatorname{tr}\bar{\bar{\mathbf{B}}}(\operatorname{tr}^2\bar{\bar{\mathbf{A}}}-\operatorname{tr}\bar{\bar{\mathbf{A}}}^2)+\operatorname{tr}\bar{\bar{\mathbf{A}}}^2\bar{\bar{\mathbf{B}}}-\operatorname{tr}\bar{\bar{\mathbf{A}}}\operatorname{tr}\bar{\bar{\mathbf{A}}}\bar{\bar{\mathbf{B}}}]n^2+\det\bar{\bar{\mathbf{A}}} = 0. \tag{53}$$

Birefringence occurs since two velocities are associated with the same direction $\bar{\mathbf{s}}$.

The polarization states can be investigated as follows: if n_1^2 and n_2^2 are two distinct, real solutions of (53), they give rise to two real directions for the complex vector $\tilde{\mathbf{e}}$, it means two linear polarization states such as

$$(\bar{\bar{\mathbf{I}}}+{}_0\bar{\bar{\mathbf{T}}}^{-1})\tilde{\mathbf{e}}_1 = -n_1^2\bar{\bar{\mathbf{S}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_1 \tag{54}$$

$$(\bar{\bar{\mathbf{I}}}+{}_0\bar{\bar{\mathbf{T}}}^{-1})\tilde{\mathbf{e}}_2 = -n_2^2\bar{\bar{\mathbf{S}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_2 \tag{55}$$

and if $n_1 \neq n_2$

$$\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_2 \,.\, (\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_1 = 0. \tag{56}$$

$\bar{\bar{\mathbf{P}}} = -\bar{\bar{\mathbf{S}}}^2$ being the projection operator on the plane perpendicular to $\bar{\mathbf{s}}$,

$$\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_2 \,.\, \bar{\bar{\mathbf{P}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{P}}}(\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_1) = 0. \tag{57}$$

In this plane perpendicular to $\bar{\mathbf{s}}$, $\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_1$ and $\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_2$ are conjugated with respect to the conic defined by

$$\bar{\bar{\mathbf{P}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{P}}}.$$

On the other hand, since $\tilde{\mathbf{b}} = n\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}$, the magnetic flux density vector is along $\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}$. The vector $\tilde{\mathbf{d}} = \tilde{\mathbf{e}}+\tilde{\mathbf{p}}$, in the plane normal to $\bar{\mathbf{s}}$, is given by

$$\tilde{\mathbf{d}} = n\bar{\bar{\mathbf{S}}}(\tilde{\mathbf{b}}-\tilde{\mathbf{m}}) = n^2\bar{\bar{\mathbf{S}}}(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}} \tag{58}$$

and owing to (56)

$$\bar{\bar{\mathbf{P}}}\tilde{\mathbf{e}}_2 \,.\, \tilde{\mathbf{d}}_1 = \bar{\bar{\mathbf{P}}}\tilde{\mathbf{e}}_1 \,.\, \tilde{\mathbf{d}}_2 = 0 \tag{59}$$

$\tilde{\mathbf{d}}_1$ is orthogonal to $\bar{\bar{\mathbf{P}}}\tilde{\mathbf{e}}_2$, and therefore parallel to $\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}_2$. The vector $\tilde{\mathbf{h}} = \tilde{\mathbf{b}}-\tilde{\mathbf{m}}$ is given by

$$\tilde{\mathbf{h}} = (\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\tilde{\mathbf{b}} = n(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}}$$

and vector $\bar{\bar{\mathbf{P}}}\tilde{\mathbf{h}}$ will be along $\bar{\bar{\mathbf{S}}}\tilde{\mathbf{d}}$, since

$$\bar{\bar{\mathbf{P}}}\tilde{\mathbf{h}} = -n\bar{\bar{\mathbf{S}}}^2(\bar{\bar{\mathbf{I}}}-{}_0\bar{\bar{\mathbf{Y}}}^{-1})\bar{\bar{\mathbf{S}}}\tilde{\mathbf{e}} = -n\bar{\bar{\mathbf{S}}}\tilde{\mathbf{d}}. \tag{60}$$

For vanishing magnetization, ${}_0\bar{\bar{\mathbf{Y}}}^{-1} = 0$, and the directions $\tilde{\mathbf{b}}_1$, $\tilde{\mathbf{b}}_2$ become orthogonal.

In the general case, the polarization modes are orthogonal if and only if the sections by the plane normal to $\bar{s}$, of the quadrics $(\bar{\bar{1}}+{}_0\bar{\bar{T}}^{-1})^{-1}$ and $(\bar{\bar{1}}-{}_0\bar{\bar{Y}}^{-1})$ have common principal directions.

4. THE MAGNETOOPTICAL EFFECTS

In this section, we analyse the effect of an external magnetic field upon a weak electromagnetic light-field in an unstrained dielectric-$[{}_0E_{AB} = 0]$. We consider only conditions which give rise to the Voigt–Cotton–Mouton effect (transverse field) and to the Faraday effect (longitudinal field).

The initial unperturbed state is characterized by

$$ {}_0x^\lambda = X^\lambda \qquad {}_0\bar{\mathbf{E}} = {}_0\bar{\mathbf{P}} = 0 \qquad {}_0\bar{\mathbf{B}} \text{ and } {}_0\bar{\mathbf{M}} \text{ uniform.} $$

The balance equations (4) and (11), and Maxwell's equations (25) to (28) are trivially satisfied, while (10) and (12) give

$$ {}_0\sigma^{kl} = {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_1}\right) g^{kl} + {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_8}\right) {}_0M^k\,{}_0M^l $$

$$ {}_0\bar{\mathbf{B}} = C_8\,{}_0\bar{\mathbf{M}} \quad \text{with} \quad C_8 = 2\rho_0\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right). \tag{61} $$

Formulae (21–23, 34, 35 and 37), together with basic representation (13) give explicit expressions for

$$ \begin{aligned} {}_0C^{\lambda\mu\alpha\nu} = {} & \rho_0\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_1^2}\right) g^{\lambda\mu} g^{\alpha\nu} + \rho_0\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_2}\right)(g^{\lambda\alpha}g^{\mu\nu} + g^{\lambda\nu}g^{\mu\alpha}) \\ & + \rho_0\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_1\,\partial I_8}\right)(g^{\lambda\mu}\,{}_0M^\alpha\,{}_0M^\nu + g^{\alpha\nu}\,{}_0M^\lambda\,{}_0M^\mu) \\ & + \tfrac{1}{2}\rho_0\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_9}\right)(g^{\lambda\nu}\,{}_0M^\mu\,{}_0M^\alpha + g^{\mu\alpha}\,{}_0M^\lambda\,{}_0M^\nu + g^{\lambda\alpha}\,{}_0M^\mu\,{}_0M^\nu + g^{\mu\nu}\,{}_0M^\lambda\,{}_0M^\alpha) \\ & + \rho_0\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_8^2}\right) {}_0M^\lambda\,{}_0M^\mu\,{}_0M^\alpha\,{}_0M^\nu \end{aligned} \tag{62} $$

$$ {}_0S_\nu^{\lambda\mu} = 0 \tag{63} $$

$$ \begin{aligned} {}_0X_\nu^{\lambda\mu} = {} & 2\,{}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_1\,\partial I_5}\right) g^{\lambda\mu}\,{}_0M_\nu + 2\,{}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_8\,\partial I_5}\right) {}_0M^\lambda\,{}_0M^\mu\,{}_0M_\nu \\ & + {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_8}\right)(\delta_\nu^\lambda\,{}_0M^\mu + {}_0M^\lambda\delta_\nu^\mu) \end{aligned} \tag{64} $$

$$ {}_0T_{\lambda\mu} = 2\,\rho_0\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_4}\right) g_{\lambda\mu} + 2\,\rho_0\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_{11}}\right) {}_0M_\lambda\,{}_0M_\mu \tag{65} $$

$$ {}_0W_{\lambda\mu} = 0 \tag{66} $$

$$ {}_0Y_{\lambda\mu} = 2\,{}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right) g_{\lambda\mu} + 4\,{}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_5^2}\right) {}_0M_\lambda\,{}_0M_\mu. \tag{67} $$

Hence, variation equations for motion, fields and constitutive relations:

$$
\begin{aligned}
{}_0\rho\ddot{u}_l = {} & C_1 u^k_{,kl} + C_2 u_{l,k}{}^k + C_3\,{}_0M^m\,{}_0M^p u_{m,pl} + (C_3-1)\,{}_0M^k\,{}_0M_l u^p_{,pk} + C_4\,{}_0M^p\,{}_0M^k u_{l,pk} \\
& + C_5\,{}_0M^m\,{}_0M_l u_m{}^{,k}{}_k + C_6\,{}_0M_p m^p_{,l} + C_7\,{}_0M_p\,{}_0M^k\,{}_0M_l m^p_{,k} + (C_4-C_5)\,{}_0M^k m_{l,k} \\
& + (C_4-C_5-C_8+1)\,{}_0M_l m^k_{,k} + \frac{1}{c}(C_8-1)(\dot{\bar{\mathbf{p}}}\times{}_0\bar{\mathbf{M}})_l \\
& - {}_0M^\mu b_{\mu,\lambda} + C_{12}\,{}_0M^m\,{}_0M^p\,{}_0M^k\,{}_0M_l u_{m,pk}
\end{aligned} \tag{68}
$$

$$\operatorname{div}\bar{\mathbf{b}} = 0 \tag{69}$$

$$\operatorname{rot}\bar{\mathbf{e}} + \frac{1}{c}\dot{\bar{\mathbf{b}}} = 0 \tag{70}$$

$$\operatorname{div}\bar{\mathbf{e}} = -\operatorname{div}\bar{\mathbf{p}} \tag{71}$$

$$\operatorname{rot}\bar{\mathbf{b}} - \frac{1}{c}\dot{\bar{\mathbf{e}}} = \operatorname{rot}\bar{\mathbf{m}} + \operatorname{rot}({}_0M^k\bar{\mathbf{u}}_{,k}) + \frac{1}{c}\dot{\bar{\mathbf{p}}} \tag{72}$$

$$\bar{\mathbf{e}} = \frac{-1}{c}(C_8+C_9)(\dot{\bar{\mathbf{u}}}\times{}_0\bar{\mathbf{M}}) + C_9\bar{\mathbf{p}} + C_{10}(\bar{\mathbf{p}}\cdot{}_0\bar{\mathbf{M}})\,{}_0\bar{\mathbf{M}} \tag{73}$$

$$
\begin{aligned}
b_l = {} & (C_6-1)\,{}_0M_l u^k_{,k} + C_7\,{}_0M^r\,{}_0M^s\,{}_0M_l u_{r,s} + (C_4-C_5+1)\,{}_0M^k u_{l,k} \\
& + (C_4-C_5+1-C_8)\,{}_0M^k u_{k,l} + C_8 m_l + C_{11}\,{}_0M_l\,{}_0M_k m^k.
\end{aligned} \tag{74}
$$

With C_α defined by

$$C_1 \equiv {}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_1^2}\right) + {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_2}\right)$$

$$C_2 \equiv {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_2}\right) + {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_1}\right)$$

$$C_3 \equiv {}_0\rho\,{}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_1\,\partial I_8}\right) + \frac{{}_0\rho}{2}\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_9}\right) - 2\,{}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right) + 1$$

$$C_4 \equiv \frac{{}_0\rho}{2}\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_9}\right) + {}_0\rho\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_8}\right) - 1$$

$$C_5 \equiv \frac{{}_0\rho}{2}\,{}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_9}\right)$$

$$C_6 \equiv 2\,{}_0\rho\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_5\,\partial I_1}\right) - 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right) + 1$$

$$C_7 \equiv 2\,{}_0\rho\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_8\,\partial I_5}\right)$$

$$C_8 \equiv 2\,{}_0\rho\,\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right)_0$$

$$C_9 \equiv 2\,{}_0\rho\,\left(\frac{\partial\tilde{\varepsilon}}{\partial I_4}\right)_0$$

$$C_{10} \equiv 2\,{}_0\rho\,\left(\frac{\partial\tilde{\varepsilon}}{\partial I_{11}}\right)_0$$

$$C_{11} \equiv 2\,{}_0\rho\,\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_5^2}\right)_0$$

$$C_{12} \equiv {}_0\rho\,\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_8^2}\right)_0.$$

Though all these invariant quantities are functions of ${}_0M^2$, one should observe that C_1, C_2 are related to the basic mechanical properties of the medium, C_9 to the electrical ones, C_8, C_{11} to the magnetic ones, while C_{10} obviously is connected with electromagnetic interaction.

(a) *The longitudinal Faraday effect*

We seek "plane wave" solutions

$$\bar{\mathbf{a}} = \operatorname{Re}\tilde{\mathbf{a}}\,e^{i\omega[t-(n/c)\bar{\mathbf{s}}\cdot\bar{\mathbf{r}}]} \tag{75}$$

of this system of equations, with $\bar{\mathbf{s}}$ (unit normal vector) parallel to ${}_0\overline{\mathbf{B}}$ and ${}_0\overline{\mathbf{M}}$.

The amplitudes are solutions of the following algebraic linear system:

$$\omega^2\left(D_1\frac{n^2}{c^2}-{}_0\rho\right)\tilde{\mathbf{u}}+\omega\left(D_2\frac{\omega n^2}{c^2}\tilde{\mathbf{u}}\bar{\mathbf{s}}+i\frac{n}{c}D_3\,{}_0M\bar{\mathbf{s}}\tilde{\mathbf{m}}\right)\bar{\mathbf{s}}$$

$$+{}_0MD_4 i\omega\frac{n}{c}\tilde{\mathbf{m}}-i\frac{\omega}{c}(C_8-1)\,{}_0M(\tilde{\mathbf{p}}\times\bar{\mathbf{s}})=0 \tag{76}$$

$$\tilde{\mathbf{e}} = C_9\tilde{\mathbf{p}}-i\frac{\omega}{c}\,{}_0M(C_8+C_9)(\tilde{\mathbf{u}}\times\bar{\mathbf{s}})+{}_0M^2C_{10}(\tilde{\mathbf{p}}\bar{\mathbf{s}})\bar{\mathbf{s}} \tag{77}$$

$$\tilde{\mathbf{b}} = {}_0M\left[(1-D_3)i\omega\frac{n}{c}\tilde{\mathbf{u}}\bar{\mathbf{s}}+{}_0MC_{11}\bar{\mathbf{s}}\tilde{\mathbf{m}}\right]\bar{\mathbf{s}}-{}_0M(D_4+1)i\omega\frac{n}{c}\tilde{\mathbf{u}}+C_8\tilde{\mathbf{m}} \tag{78}$$

$$\tilde{\mathbf{b}} = n\bar{\mathbf{s}}\times\tilde{\mathbf{e}} \tag{79}$$

$$\tilde{\mathbf{e}} = n\tilde{\mathbf{b}}\times\bar{\mathbf{s}}-\tilde{\mathbf{p}}-n\tilde{\mathbf{m}}\times\bar{\mathbf{s}}-i\,{}_0M\frac{\omega}{c}n^2\bar{\mathbf{s}}\times\tilde{\mathbf{u}} \tag{80}$$

with

$$D_1 = C_2+{}_0M^2C_4$$

$$D_2 = C_1+2C_3\,{}_0M^2-{}_0M^2+C_5\,{}_0M^2+C_{12}\,{}_0M^4$$

$$D_3 = C_6+{}_0M^2C_7+C_4-C_5-C_8+1$$

$$D_4 = C_4-C_5.$$

It can be solved in the following way.

Elimination of $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{b}}$ between (77) to (80), gives two vector-valued equations for the unknowns $\tilde{\mathbf{u}}, \tilde{\mathbf{p}}, \tilde{\mathbf{m}}$

$$\left[{}_0M(1-D_3-C_8-C_9)i\omega\frac{n}{c}\tilde{\mathbf{u}}\bar{\mathbf{s}}+{}_0M^2C_{11}\bar{\mathbf{s}}\tilde{\mathbf{m}}\right]\bar{\mathbf{s}}$$

$$-{}_0M(D_4+1-C_8-C_9)i\omega\frac{n}{c}\tilde{\mathbf{u}}+C_8\tilde{\mathbf{m}}-nC_9\bar{\mathbf{s}}\times\tilde{\mathbf{p}}=0$$

$$(1+C_9)\tilde{\mathbf{p}}-i\frac{\omega}{c}{}_0M(C_8+C_9-n^2D_4)\tilde{\mathbf{u}}\times\bar{\mathbf{s}}+{}_0M^2C_{10}(\tilde{\mathbf{p}}\bar{\mathbf{s}})\bar{\mathbf{s}}+n(1-C_8)\tilde{\mathbf{m}}\times\bar{\mathbf{s}}=0. \tag{82}$$

We now take the scalar and vector product of (76), (81) and (82) with $\bar{\mathbf{s}}$

$$\omega^2\left[(D_1+D_2)\frac{n^2}{c^2}-{}_0\rho\right]\tilde{\mathbf{u}}\bar{\mathbf{s}}+i\omega\frac{n}{c}{}_0M(D_3+D_4)\tilde{\mathbf{m}}\bar{\mathbf{s}}=0 \tag{83}$$

$$-i\omega\frac{n}{c}{}_0M(D_3+D_4)\tilde{\mathbf{u}}\bar{\mathbf{s}}+({}_0M^2C_{11}+C_8)\tilde{\mathbf{m}}\bar{\mathbf{s}}=0 \tag{84}$$

$$(1+C_9+{}_0M^2C_{10})\tilde{\mathbf{p}}\bar{\mathbf{s}}=0. \tag{85}$$

Hence, $\tilde{\mathbf{p}}\bar{\mathbf{s}}=0$, and $\tilde{\mathbf{u}}\bar{\mathbf{s}}=\tilde{\mathbf{m}}\bar{\mathbf{s}}=0$, except when

$$\frac{v^2}{c^2}\equiv\frac{1}{n^2}=\frac{D_1+D_2}{{}_0\rho c^2}\left[1-\frac{{}_0M^2(D_3+D_4)^2}{(D_1+D_2)({}_0M^2C_{11}+C_8)}\right] \tag{86}$$

which gives the squared velocity v^2 of the longitudinal elastic wave.

$$\omega^2\left(D_1\frac{n^2}{c^2}-{}_0\rho\right)\tilde{\mathbf{u}}\times\bar{\mathbf{s}}+i\omega\frac{n}{c}{}_0MD_4\tilde{\mathbf{m}}\times\bar{\mathbf{s}}+i\frac{\omega}{c}(C_8-1)\,{}_0M\tilde{\mathbf{p}}=0 \tag{87}$$

$$-{}_0M(D_4+1-C_8-C_9)i\omega\frac{n}{c}\tilde{\mathbf{u}}\times\bar{\mathbf{s}}+C_8\tilde{\mathbf{m}}\times\bar{\mathbf{s}}-nC_9\tilde{\mathbf{p}}=0 \tag{88}$$

$$-{}_0M\frac{i\omega}{c}(C_8+C_9-D_4n^2)\tilde{\mathbf{u}}\times\bar{\mathbf{s}}+n(1-C_8)\tilde{\mathbf{m}}\times\bar{\mathbf{s}}+(1+C_9)\tilde{\mathbf{p}}=0. \tag{89}$$

This system has nontrivial solutions only when

$$\begin{vmatrix} \mu+C_4\lambda-\beta^2 & \frac{i}{c}{}_0M(C_4-C_5) & \frac{i}{c}{}_0M(C_8-1)\beta \\ -\frac{i}{c}(C_4-C_5+1-C_8-C_9)\frac{{}_0M}{{}_0\rho} & C_8\beta & -C_9 \\ -\frac{i}{c}[(C_8+C_9)\beta^2-C_4+C_5)]\frac{{}_0M}{{}_0\rho}\beta & 1-C_8 & (1+C_9)\beta \end{vmatrix}=0 \tag{90}$$

with

$$\beta=\frac{1}{n}\qquad\mu=\frac{C_2}{{}_0\rho c^2}\qquad\lambda=\frac{{}_0M^2}{{}_0\rho c^2}.$$

The coefficients of this second degree equation in β, depend upon dimensionless quantities λ and μ, which may be regarded as small parameters. Indeed, balance equations used at the starting point, are only valid in the nonrelativistic approximation: $\bar{\mathbf{B}}\,.\,\bar{\mathbf{M}}/\rho c^2$ and $\bar{\mathbf{E}}\,.\,\bar{\mathbf{P}}/\rho c^2$ were considered as small quantities and that implies that ${}_0M^2/{}_0\rho c^2$ is small in the case of weak magnetization. On the other hand, the ratio of the velocity of the transverse sound wave to the velocity of light may also be considered as a small quantity. The roots (when computation is restricted to the lowest order in λ, μ) are

$$\frac{1}{n^2} = \frac{C_2}{{}_0\rho c^2}\left[1+\left\{\frac{C_4}{C_2}+\frac{(C_4-C_5)^2}{C_2(1-C_8)}\right\}{}_0M^2\right] \tag{91}$$

$$\frac{1}{n^2} = \frac{C_9}{1+C_9}\,\frac{C_8-1}{C_8}. \tag{92}$$

The former wave is a transverse elastic wave, the latter a transverse electromagnetic light wave. Both have arbitrary polarization states. (92) is the classical expression of light velocity as a function of the dielectric and magnetic permittivities, C_9 and C_8 depending on ${}_0M^2$, hence on ${}_0B^2$. For nonmagnetizable dielectric, ${}_0\bar{\mathbf{M}} = 0$, $1/C_8 \to 0$, the 3 roots for β^2 (86, 91 and 92) reduce to the classical values

$$\frac{C_1(0)+C_2(0)}{{}_0\rho c^2},\qquad \frac{C_2(0)}{{}_0\rho c^2},\qquad \frac{C_9(0)}{1+C_9(0)}$$

$C_1(0)$, $C_2(0)$ being Lamé's parameters, $1/C_9(0)$ the dielectric susceptibility of the medium.

In short, three kinds of waves may propagate in the medium: two elastic (slow) waves, respectively longitudinal and transverse, and one transverse electromagnetic (fast) wave. No qualitative change appears as a consequence of the magnetization of the medium.

In particular, this model of elastic polarizable and magnetizable dielectric medium, *does not* exhibit any Faraday rotation.

(b) *The transverse birefringence effect*

The same procedure is applied to waves with vector $\bar{\mathbf{s}}$ normal to ${}_0\bar{\mathbf{B}}$ and ${}_0\bar{\mathbf{M}}$. The equations for amplitudes:

$$\omega^2\left(C_2\frac{n^2}{c^2}-{}_0\rho\right)\tilde{\mathbf{u}}+\frac{\omega n}{c}\left[C_1\omega\frac{n}{c}\tilde{\mathbf{u}}\bar{\mathbf{s}}+iC_6\tilde{\mathbf{m}}\,{}_0\bar{\mathbf{M}}-i\tilde{\mathbf{b}}\,{}_0\bar{\mathbf{M}}\right]\bar{\mathbf{s}}$$

$$+\omega\frac{n}{c}\left[C_5\omega\frac{n}{c}\tilde{\mathbf{u}}\,{}_0\bar{\mathbf{M}}+i(C_4-C_5-C_8+1)\tilde{\mathbf{m}}\bar{\mathbf{s}}\right]{}_0\bar{\mathbf{M}}-\frac{i\omega}{c}(C_8-1)\tilde{\mathbf{p}}\times{}_0\bar{\mathbf{M}} = 0 \tag{93}$$

$$\tilde{\mathbf{e}} = C_9\tilde{\mathbf{p}}+C_{10}(\tilde{\mathbf{p}}\,{}_0\bar{\mathbf{M}})\,{}_0\bar{\mathbf{M}}-i\frac{\omega}{c}(C_8+C_9)(\tilde{\mathbf{u}}\times{}_0\bar{\mathbf{M}}) \tag{94}$$

$$\tilde{\mathbf{b}} = \left[C_{11}\tilde{\mathbf{m}}\,.\,{}_0\bar{\mathbf{M}}-i\omega\frac{n}{c}(C_6-1)\tilde{\mathbf{u}}\bar{\mathbf{s}}\right]{}_0\bar{\mathbf{M}}-i\omega\frac{n}{c}\,.\,(C_4-C_5+1-C_8)(\tilde{\mathbf{u}}\,{}_0\bar{\mathbf{M}})\bar{\mathbf{s}}+C_8\tilde{\mathbf{m}} \tag{95}$$

$$\tilde{\mathbf{b}} = n\bar{\mathbf{s}}\times\tilde{\mathbf{e}} \tag{96}$$

$$\tilde{\mathbf{e}} = n\tilde{\mathbf{b}}\times\bar{\mathbf{s}}-\tilde{\mathbf{p}}-n\tilde{\mathbf{m}}\times\bar{\mathbf{s}} \tag{97}$$

can be solved by elimination of $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{b}}$, and projection of the 3 obtained vector-valued equations on $\bar{\mathbf{s}}$, ${}_0\overline{\mathbf{M}}$, $\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}}$.

$$\omega^2\left[\frac{n^2}{c^2}(C_1+C_2-C_6\,{}_0M^2+{}_0M^2)-{}_0\rho\right]\tilde{\mathbf{u}}\bar{\mathbf{s}}+i\omega\frac{n}{c}(C_6-C_8-C_{11}\,{}_0M^2)\tilde{\mathbf{m}}\,{}_0\overline{\mathbf{M}}$$

$$-i\frac{\omega}{c}(C_8-1)\tilde{\mathbf{p}}\,.\,{}_0\overline{\mathbf{M}}\times\bar{\mathbf{s}}=0 \tag{98}$$

$$\omega^2\left[\frac{n^2}{c^2}(C_2+{}_0M^2C_5)-{}_0\rho\right]\tilde{\mathbf{u}}\,{}_0\overline{\mathbf{M}}+i\omega\frac{n}{c}\,{}_0M^2(C_4-C_5-C_8+1)\tilde{\mathbf{m}}\bar{\mathbf{s}}=0 \tag{99}$$

$$\omega^2\left[C_2\frac{n^2}{c^2}-{}_0\rho\right]\tilde{\mathbf{u}}\,.\,\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}}-i\frac{\omega}{c}(C_8-1)\,{}_0M^2\tilde{\mathbf{p}}\bar{\mathbf{s}}=0 \tag{100}$$

$$-i\omega\frac{n}{c}(C_4-C_5+1-C_8)\tilde{\mathbf{u}}\,{}_0\overline{\mathbf{M}}+C_8\tilde{\mathbf{m}}\bar{\mathbf{s}}=0 \tag{101}$$

$$({}_0M^2C_{11}+C_8)\tilde{\mathbf{m}}\,{}_0\overline{\mathbf{M}}-i\omega\frac{n}{c}(C_6-1+C_8+C_9)\,{}_0M^2\tilde{\mathbf{u}}\bar{\mathbf{s}}-nC_9\tilde{\mathbf{p}}\,.\,{}_0\overline{\mathbf{M}}\times\bar{\mathbf{s}}=0 \tag{102}$$

$$C_8\tilde{\mathbf{m}}(\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}})-n(C_9+{}_0M^2C_{10})\tilde{\mathbf{p}}\,{}_0\overline{\mathbf{M}}=0 \tag{103}$$

$$(1+C_9)\tilde{\mathbf{p}}\bar{\mathbf{s}}-i\frac{\omega}{c}(C_8+C_9)\tilde{\mathbf{u}}\,.\,{}_0\overline{\mathbf{M}}\times\bar{\mathbf{s}}=0 \tag{104}$$

$$n(C_8-1)\tilde{\mathbf{m}}\,.\,(\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}})-(1+C_9+C_{10}\,{}_0M^2)\tilde{\mathbf{p}}\,{}_0\overline{\mathbf{M}}=0 \tag{105}$$

$$-n(C_{11}\,{}_0M^2+C_8-1)\tilde{\mathbf{m}}\,{}_0\overline{\mathbf{M}}+\frac{i\omega}{c}\,{}_0M^2[(C_6-1)n^2+C_8+C_9]\tilde{\mathbf{u}}\bar{\mathbf{s}}-(1+C_9)\tilde{\mathbf{p}}\,.\,\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}}=0 \tag{106}$$

From (103), (105), $\tilde{\mathbf{p}}\,{}_0\overline{\mathbf{M}}=\tilde{\mathbf{m}}\,.\,\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}}=0$, except when

$$\frac{1}{n^2}=\frac{C_8-1}{C_8}\,\frac{C_9+{}_0M^2C_{10}}{1+C_9+{}_0M^2C_{10}}. \tag{107}$$

In the same way, (100) (104) and (99) (101) (when computation is restricted to the lowest order terms in λ and μ) imply respectively

$$\frac{1}{n_2}=\frac{C_2}{{}_0\rho c^2}\qquad\text{or } \tilde{\mathbf{u}}\,.\,\bar{\mathbf{s}}\times{}_0\overline{\mathbf{M}}=\tilde{\mathbf{p}}\bar{\mathbf{s}}=0 \tag{108}$$

and

$$\frac{1}{n^2}=\frac{C_2}{{}_0\rho c^2}\left[1+\frac{C_5}{C_2}\,{}_0M^2-\frac{(C_4-C_5-C_8+1)^2}{C_2C_8}\,{}_0M^2\right] \tag{109}$$

or

$$\tilde{\mathbf{u}}\,{}_0\overline{\mathbf{M}}=\tilde{\mathbf{m}}\bar{\mathbf{s}}=0.$$

Finally, (98, 102 and 106) admit nontrivial solutions only when the characteristic determinant of the system is zero.

The latter gives a biquadratic equation for $1/n$, whose solutions, limited to the slowest order in the small parameters λ, μ and $v \equiv (C_1+C_2)/_0\rho c^2$, are

$$\frac{1}{n^2} = \frac{C_1+C_2}{_0\rho c^2}\left[1-\frac{(1-C_6)^2\,{}_0M^2}{(C_1+C_2)({}_0M^2C_{11}+C_8-1)}\right] \tag{110}$$

$$\frac{1}{n^2} = \frac{C_9}{(C_9+1)}\cdot\frac{({}_0M^2C_{11}+C_8-1)}{({}_0M^2C_{11}+C_8)}. \tag{111}$$

Discussion of results (107) to (111) leads to some important conclusions. For any direction $\bar{\mathbf{s}}$, only the following propagation modes are permissible:

(1) *One single* longitudinal (slow) wave, the speed of which given by (110) reduces to the classical Lamé value when magnetization vanishes.

(2) *Two* transverse elastic (slow) waves, with velocities given by (108) and (109), the former with polarization state along $\bar{\mathbf{s}} \times {}_0\bar{\mathbf{B}}$, the latter with polarization state along ${}_0\bar{\mathbf{B}}$.

The medium exhibits thus *acoustical transverse birefringence*, which vanishes with magnetization, reducing in this case to the classical transverse Lamé value.

(3) *Two* transverse electromagnetic (fast) waves, with velocities given by (107) and (111), with polarization along and normal to the external applied field ${}_0\bar{\mathbf{B}}$.

The medium exhibits *optical transverse linear birefringence* and gives account for the Voigt–Cotton–Mouton effect. This birefringence vanishes also with the magnetization.

The authors think that this result, obviously due to the inclusion of magnetization in the balance and constitutive equations, is an original one.

5. THE ELECTROOPTICAL EFFECTS

The electrooptical Kerr effect can be derived from the classical model of polarizable but *nonmagnetizable* elastic dielectric described by Toupin [1]. Using the previous method, one can investigate the influence of the magnetization on this effect.

The initial nonperturbed state in this case:

$${}_0x^\lambda = X^\lambda \qquad {}_0\bar{\mathbf{E}} \neq 0 \qquad {}_0\bar{\mathbf{P}} \neq 0 \text{ uniform } {}_0\bar{\mathbf{B}} = {}_0\bar{\mathbf{M}} = 0.$$

Equations (4) and (12) and Maxwell's equations are trivially satisfied, while (10) and (11) gives

$${}_0\sigma^{kl} = {}_0\rho\,{}_0\left(\frac{\partial\bar{\varepsilon}}{\partial I_1}\right)g^{kl} + {}_0\rho\,{}_0\left(\frac{\partial\bar{\varepsilon}}{\partial I_6}\right){}_0P^k\,{}_0P^l$$

and

$${}_0\bar{\mathbf{E}} = C_9\,{}_0\bar{\mathbf{P}} \quad \text{with} \quad C_9 = 2\,{}_0\rho\left(\frac{\partial\bar{\varepsilon}}{\partial I_4}\right) \tag{112}$$

$1/C_9$ is the dielectric permittivity.

Again, from formulae (21–23, 34, 35 and 37) and basic representation (13)

$$
\begin{aligned}
{}_0C^{klmp} &= {}_0\rho \left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_1^2}\right)_0 g^{kl}g^{mp} + {}_0\rho \left(\frac{\partial\tilde{\varepsilon}}{\partial I_2}\right)_0 (g^{lm}g^{kp}+g^{lp}g^{km}) \\
&+ {}_0\rho \left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_6\,\partial I_1}\right)_0 (g^{lk}\,{}_0P^m\,{}_0P^p + g^{mp}\,{}_0P^l\,{}_0P^k) \\
&+ \tfrac{1}{2}\,{}_0\rho \left(\frac{\partial\tilde{\varepsilon}}{\partial I_7}\right)_0 (g^{lm}\,{}_0P^k\,{}_0P^p + g^{lp}\,{}_0P^k\,{}_0P^p + g^{km}\,{}_0P^l\,{}_0P^p + g^{kp}\,{}_0P^l\,{}_0P^m) \\
&+ {}_0\rho \left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_6^2}\right)\,{}_0P^l\,{}_0P^k\,{}_0P^m\,{}_0P^p
\end{aligned} \tag{113}
$$

$$
{}_0S_r^{lk} = 2\,{}_0\rho\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_4\,\partial I_1}\right) g^{lk}\,{}_0P_r + 2\,{}_0\rho\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_4\,\partial I_6}\right){}_0P^l\,{}_0P^k\,{}_0P_r + {}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_6}\right)(\delta_r^l\,{}_0P^k + \delta_r^k\,{}_0P^l) \tag{114}
$$

$$
{}_0X_r^{lk} = \frac{{}_0\rho}{2}\left(\frac{\partial\tilde{\varepsilon}}{\partial I_{12}}\right)(\varepsilon^{lqs}\,{}_0P^k\,{}_0P_s g_{qr} + \varepsilon^{kqs}\,{}_0P^l\,{}_0P_s g_{qr}) \tag{115}
$$

$$
{}_0T_{kl} = 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_4}\right)_0 g_{kl} + 4\,{}_0\rho\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_4^2}\right)_0 {}_0P_k\,{}_0P_l \tag{116}
$$

$$
{}_0W_{kl} = 0 \tag{117}
$$

$$
{}_0Y_{kl} = 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_5}\right)_0 g_{kl} + 2\,{}_0\rho\left(\frac{\partial\tilde{\varepsilon}}{\partial I_{11}}\right)_0 {}_0P_k\,{}_0P_l. \tag{118}
$$

Variation equations (15, 29–33 and 36) become

$$
\begin{aligned}
{}_0\rho\ddot{u}_l &= C_1 u^m_{,ml} + C_2 u_l{}^{,m}{}_m + D_3\,{}_0P^m\,{}_0P^p u_{m,pl} + D_3\,{}_0P^k\,{}_0P_l u^m_{,km} + (D_4+D_7)\,{}_0P^p\,{}_0P^k u_{l,pk} \\
&+ D_4\,{}_0P^m\,{}_0P_l u_m{}^{,k}{}_k + D_{11}\,{}_0P^k\,{}_0P^m\,{}_0P^p u_{m,pk}\,{}_0P_l + D_5\,{}_0P^m p_{m,l} + D_6\,{}_0P_l\,{}_0P^k\,{}_0P^m p_{m,k} \\
&+ D_7\,{}_0P^k p_{l,k} + (D_7 - C_9)\,{}_0P_l p^k_{,k} + D_8\Big(\varepsilon_{lrs}\,{}_0P^s\,{}_0P^k m^r{}_{,k} - \frac{1}{c}\,{}_0P^2\,{}_0P^k\dot{u}_{l,k} \\
&+ \frac{2}{c}\,{}_0P_l\,{}_0P^r\,{}_0P^k\dot{u}_{r,k} + \varepsilon^{krs}\,{}_0P_s\,{}_0P_l m_{r,k} - \frac{1}{c}\,{}_0P_l\,{}_0P^2\dot{u}^k_{,k}\Big) + \frac{1}{c}C_9({}_0\mathbf{\bar{P}}\times\dot{\mathbf{\bar{m}}})_l
\end{aligned} \tag{119}
$$

$$
\operatorname{div}\mathbf{\bar{b}} = 0 \tag{120}
$$

$$
\operatorname{rot}\mathbf{\bar{e}} + \frac{1}{c}\dot{\mathbf{\bar{b}}} = 0 \tag{121}
$$

$$
\operatorname{div}\mathbf{\bar{e}} = -\operatorname{div}\mathbf{\bar{p}} \tag{122}
$$

$$
\operatorname{rot}\mathbf{\bar{b}} - \frac{1}{c}\dot{\mathbf{\bar{e}}} = \operatorname{rot}\mathbf{\bar{m}} + \frac{1}{c}\dot{\mathbf{\bar{p}}} - \frac{1}{c}\operatorname{div}\dot{\mathbf{\bar{u}}}\,{}_0\mathbf{\bar{P}} + \frac{1}{c}\,{}_0P^k\dot{\mathbf{\bar{u}}}_{,k} \tag{123}
$$

$$
\begin{aligned}
e_l &= D_5 \operatorname{div}\mathbf{\bar{u}}\,{}_0P_l + D_6\,{}_0P_l\,{}_0P^r\,{}_0P^s u_{r,s} + D_7(u_{l,k}\,{}_0P^k + u_{k,l}\,{}_0P^k) + C_9 p_l \\
&+ D_{12}({}_0P_k p^k)\,{}_0P_l - C_9\,{}_0P_k u^k_{,l}
\end{aligned} \tag{124}
$$

$$b_l = \frac{1}{c}C_9(\bar{\mathbf{u}} \times {}_0\bar{\mathbf{P}})_l + D_8(\varepsilon^{qrs}\, {}_0P^k\, {}_0P_s g_{lr} + \varepsilon^{krs}\, {}_0P^q\, {}_0P_s g_{rl})u_{q,k}$$

$$+ C_8\left[m_l + \frac{1}{c}(\dot{\bar{\mathbf{u}}} \times {}_0\bar{\mathbf{P}})_l\right] + C_{10}\, {}_0P_l\, {}_0P_k m^k \qquad (125)$$

with

$$D_3 = {}_0\rho\, {}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_6\, \partial I_1}\right) + \frac{1}{2}\, {}_0\rho\, {}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_7}\right)$$

$$D_4 = \frac{1}{2}\, {}_0\rho\, {}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_7}\right)$$

$$D_5 = 2\, {}_0\rho\, {}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_4\, \partial I_1}\right)$$

$$D_6 = 2\, {}_0\rho\, {}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_4\, \partial I_6}\right)$$

$$D_7 = {}_0\rho\, {}_0\!\left(\frac{\partial\tilde{\varepsilon}}{\partial I_6}\right)$$

$$D_8 = \frac{1}{2}\, {}_0\rho\, \left(\frac{\partial\tilde{\varepsilon}}{\partial I_{12}}\right)$$

$$D_{11} = {}_0\rho\, {}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_6^2}\right)$$

$$D_{12} = 4\, {}_0\rho\, {}_0\!\left(\frac{\partial^2\tilde{\varepsilon}}{\partial I_4^2}\right).$$

All invariants C_α and D_α depend on ${}_0P^2$, and as a rule, have different values than the quantities introduced in the preceding section.

(a) *The transverse Kerr effect*

We seek "plane wave" solutions, with $\bar{\mathbf{s}}$ perpendicular to ${}_0\mathbf{E}$ and ${}_0\bar{\mathbf{P}}$. Hence,

$$\omega^2\left(C_2\frac{n^2}{c^2} - {}_0\rho\right)\tilde{\mathbf{u}} + \omega\frac{n}{c}\left(C_1\omega\frac{n}{c}\tilde{\mathbf{u}}\bar{\mathbf{s}} + iD_5\, {}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{p}}\right)\bar{\mathbf{s}}$$

$$+\omega\frac{n}{c}\left[D_4\omega\frac{n}{c}\, {}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{u}} + i(D_7 - C_9)\tilde{\mathbf{p}}\,.\,\bar{\mathbf{s}} + iD_8\bar{\mathbf{s}}\,.\,\tilde{\mathbf{m}} \times {}_0\bar{\mathbf{P}}\right.$$

$$\left. + D_8\frac{\omega}{c}\, {}_0P^2\tilde{\mathbf{u}}\bar{\mathbf{s}}\right]{}_0\bar{\mathbf{P}} - i\frac{\omega}{c}C_9\, {}_0\bar{\mathbf{P}} \times \tilde{\mathbf{m}} = 0 \qquad (126)$$

$$\tilde{\mathbf{e}} = \left(D_{12}\, {}_0\bar{\mathbf{P}}\tilde{\mathbf{p}} - i\omega\frac{n}{c}D_5\tilde{\mathbf{u}}\bar{\mathbf{s}}\right){}_0\bar{\mathbf{P}} - (D_7 - C_9)i\omega\frac{n}{c}({}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{u}})\bar{\mathbf{s}} + C_9\tilde{\mathbf{p}}. \qquad (127)$$

$$\tilde{\mathbf{b}} = \frac{i\omega}{c}(C_8 + C_9)\tilde{\mathbf{u}} \times {}_0\bar{\mathbf{P}} - i\omega\frac{n}{c}D_8({}_0\bar{\mathbf{P}}\tilde{\mathbf{u}})\, {}_0\bar{\mathbf{P}} \times \bar{\mathbf{s}} + C_8\tilde{\mathbf{m}} + C_{10}({}_0\bar{\mathbf{P}}\tilde{\mathbf{m}})\, {}_0\bar{\mathbf{P}} \qquad (128)$$

$$\tilde{\mathbf{b}} = n\bar{\mathbf{s}} \times \tilde{\mathbf{e}} \tag{129}$$

$$n\tilde{\mathbf{b}} \times \bar{\mathbf{s}} - \tilde{\mathbf{e}} = \tilde{\mathbf{p}} + n\tilde{\mathbf{m}} \times \bar{\mathbf{s}} + i\frac{\omega}{c}n(\bar{\mathbf{s}}\tilde{\mathbf{u}})\,{}_0\bar{\mathbf{P}}. \tag{130}$$

Elimination of $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{b}}$ can be carried out in order to obtain 3 vectorvalued equations in $\tilde{\mathbf{u}}$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{m}}$, which are projected on the directions $\bar{\mathbf{s}}$, ${}_0\bar{\mathbf{P}}$, $\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}}$

$$\omega\left[(C_1+C_2)\frac{n^2}{c^2} - {}_0\rho\right]\tilde{\mathbf{u}}\bar{\mathbf{s}} + i\frac{n}{c}D_5\,{}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{p}} - \frac{i}{c}C_9\bar{\mathbf{s}}\,.\,{}_0\bar{\mathbf{P}} \times \tilde{\mathbf{m}} = 0 \tag{131}$$

$$\omega\left[(C_2 + D_4\,{}_0P^2)\frac{n^2}{c^2} - {}_0\rho\right]\tilde{\mathbf{u}}\,{}_0\bar{\mathbf{P}} + i\frac{n}{c}(D_7 - C_8)\,{}_0P^2\tilde{\mathbf{p}}\bar{\mathbf{s}}$$

$$+ i\frac{n}{c}D_8\,{}_0P^2\bar{\mathbf{s}}\,.\,\tilde{\mathbf{m}} \times {}_0\bar{\mathbf{P}} + n\frac{\omega}{c^2}D_8\,{}_0P^4\tilde{\mathbf{u}}\bar{\mathbf{s}} = 0 \tag{132}$$

$$\omega\left(C_2\frac{n^2}{c^2} - {}_0\rho\right)\tilde{\mathbf{u}}\,.\,\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}} + \frac{i}{c}C_9\,{}_0P^2\tilde{\mathbf{m}}\bar{\mathbf{s}} = 0 \tag{133}$$

$$\frac{i\omega}{c}(C_8 + C_9)\bar{\mathbf{s}}\,.\,\tilde{\mathbf{u}} \times {}_0\bar{\mathbf{P}} + C_8\tilde{\mathbf{m}}\bar{\mathbf{s}} = 0 \tag{134}$$

$$(C_8 + C_{10}\,{}_0P^2)\tilde{\mathbf{m}}\,{}_0\bar{\mathbf{P}} - nC_9\,{}_0\bar{\mathbf{P}}\bar{\mathbf{s}} \times \tilde{\mathbf{p}} = 0 \tag{135}$$

$$\frac{i\omega}{c}\,{}_0P^2[C_8 + C_9 + D_5n^2] + i\omega\frac{n}{c}D_8(\tilde{\mathbf{u}}\,{}_0\bar{\mathbf{P}})\,{}_0P^2$$

$$+ C_8\tilde{\mathbf{m}}\,.\,\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}} - n(D_{12}\,{}_0P^2 + C_9)\tilde{\mathbf{p}}\,{}_0\bar{\mathbf{P}} = 0 \tag{136}$$

$$i\omega\frac{n}{c}(D_7 - C_9)\,{}_0\bar{\mathbf{P}}\tilde{\mathbf{u}} - (C_9 + 1)\tilde{\mathbf{p}}\bar{\mathbf{s}} = 0 \tag{137}$$

$$i\omega\frac{n}{c}(C_8 + C_9 + D_5 - 1)\,{}_0P^2\tilde{\mathbf{u}}\bar{\mathbf{s}} + i\omega\frac{n^2}{c}D_8\,{}_0P^2({}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{u}})$$

$$-(D_{12}\,{}_0P^2 + C_9 + 1)\,{}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{p}} + n(C_8 - 1)\,{}_0\bar{\mathbf{P}}\,.\,\tilde{\mathbf{m}} \times \bar{\mathbf{s}} = 0 \tag{138}$$

$$n(C_8 - 1 + C_{10}\,{}_0P^2)\tilde{\mathbf{m}}\,{}_0\bar{\mathbf{P}} + (C_9 + 1)\tilde{\mathbf{p}}\,.\,\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}} = 0. \tag{139}$$

We further consider the dimensionless parameters

$$\mu = \frac{C_2}{{}_0\rho c^2} \qquad \nu = \frac{C_1 + C_2}{{}_0\rho c^2} \qquad \lambda = \frac{{}_0P^2}{{}_0\rho c^2}$$

as small quantities.

From (133–135 and 139), at the lowest order in μ, λ, ν, we obtain:

$$\frac{1}{n^2} = \frac{C_2}{{}_0\rho c^2}\ \text{(slow wave)} \quad \text{or} \quad \tilde{\mathbf{u}}\,.\,\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}} = \tilde{\mathbf{m}}\bar{\mathbf{s}} = 0 \tag{140}$$

$$\frac{1}{n^2} = \frac{C_9(C_8 - 1 + C_{10}\,{}_0P^2)}{(C_9 + 1)(C_8 + C_{10}\,{}_0P^2)}\ \text{(fast wave)} \quad \text{or} \quad \tilde{\mathbf{m}}\,{}_0\bar{\mathbf{P}} = {}_0\bar{\mathbf{P}}\,.\,\bar{\mathbf{s}} \times \tilde{\mathbf{p}} = 0 \tag{141}$$

Elimination of $\bar{\mathbf{p}}\bar{\mathbf{s}}$ between (137) and (132), and of $\bar{\mathbf{s}}\,.\,{}_0\bar{\mathbf{P}}x\bar{\mathbf{m}}$ between (131, 132, 136 and 138) gives a linear system which admits a nontrivial solution only when its characteristic determinant vanishes.

$$\begin{vmatrix} C_8\nu - C_9D_5\lambda + [C_8+C_9(C_8+C_9)\lambda]\beta^2 & i(C_8D_5 - C_9^2D_{12\,0}P^2) & -C_9D_8\lambda \\ -D_{8\,0}P^2(C_9+D_5)\lambda\beta^2 & -iD_{8\,0}P^2(D_{12\,0}P^2+C_9+1)\beta^2 & (C_8-1)\mu+\left[D_4(C_8-1) - \dfrac{(D_7-C_9)^2(C_8-1)}{C_9+1} - D_{8\,0}^2P^2\right]\lambda - (C_8-1)\beta^2 \\ i[D_5(C_8-1)-(C_9+C_8D_5)\beta^2] & [C_8D_{12\,0}P^2+C_8(C_9+1)]\beta^2 - (C_8-1)(D_{12\,0}P^2+C_9) & -iD_8\lambda \end{vmatrix} = 0 \quad (142)$$

This algebraic equation of the third degree in $\beta^2 = 1/n^2$ admits solutions of the form

$$\beta^2 = {}_0x + {}_1x\lambda + {}_2x\mu + {}_3x\nu + \ldots$$

By substitution, one obtains a solution of zero order

$$\frac{1}{n^2} = \frac{(C_8-1)(D_{12\,0}P^2+C_9)}{C_8(D_{12\,0}P^2+C_9+1)} \quad \text{(fast wave)} \qquad (144)$$

and two solutions of order one (slow waves)

$$\frac{1}{n^2} = \frac{C_1+C_2}{{}_0\rho c^2}\left[1 - \frac{D_5^2\,{}_0P^2}{(C_1+C_2)(D_{12\,0}P^2+C_9)}\right] \qquad (145)$$

$$\frac{1}{n^2} = \frac{C_2}{{}_0\rho c^2}\left[1 + \left\{D_4 - \frac{(D_7-C_9)^2}{C_9+1} - \frac{D_8^2\,{}_0P^2}{C_8-1}\right\}\frac{{}_0P^2}{C_2}\right]. \qquad (146)$$

Investigation of the homogeneous system leads to the following results (at the same approximation): (145) is an acoustic longitudinal wave, (140) a transverse sound wave with vibration along $\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}}$, (146) a sound wave with polarization perpendicular to the latter, but only transverse when the magnetization is small ($1/C_8 = 0(\lambda)$). (141) is a transverse light wave with polarization along $\bar{\mathbf{s}} \times {}_0\bar{\mathbf{P}}$, (144) a light wave with polarization vector perpendicular to the latter, and, again, truly transverse only for small magnetization.

The continuum model involves the Kerr effect, as did the nonmagnetizable model investigated by Toupin [1]. The magnetization brings only quantitative changes in the velocities, but no new phenomena. For vanishing polarization, the Kerr effect vanishes also, as the Cotton–Mouton effect did for vanishing magnetization.

(b) *The longitudinal Pockels effect*

Seeking solutions with wave vector along the electrical field direction, one finds

$$\omega^2\left[\{C_2+{}_0P^2(D_4+D_7)\}\frac{n^2}{c^2}+D_8\,{}_0P^3\frac{n}{c^2}-{}_0\rho\right]\bar{\mathbf{u}}$$

$$+\{[(C_1+2D_3\,{}_0P^2+D_4\,{}_0P^2+D_{11}\,{}_0P^4)n-D_8\,{}_0P^3]\bar{\mathbf{u}}\bar{\mathbf{s}}$$

$$+i\omega\frac{n}{c}\,{}_0P(D_7-C_9+D_5+D_6\,{}_0P^2)\bar{\mathbf{p}}\bar{\mathbf{s}}\}\bar{\mathbf{s}}$$

$$+i\omega\frac{n}{c}D_7\,{}_0P\bar{\mathbf{p}}+i\omega\frac{n}{c}\,{}_0P^2D_8\bar{\mathbf{m}}\times\bar{\mathbf{s}}-\frac{i\omega}{c}C_9\,{}_0P\bar{\mathbf{s}}\times\bar{\mathbf{m}}=0 \qquad (147)$$

$$\bar{\mathbf{e}}={}_0P\left[-i\omega\frac{n}{c}(D_5+D_6\,{}_0P^2)\bar{\mathbf{u}}\bar{\mathbf{s}}+D_{12}\,{}_0P\bar{\mathbf{p}}\bar{\mathbf{s}}\right]\bar{\mathbf{s}}$$

$$-i\omega\frac{n}{c}D_7\,{}_0P\bar{\mathbf{u}}-i\omega\frac{n}{c}(D_7-C_9)\,{}_0P(\bar{\mathbf{s}}\bar{\mathbf{u}})\bar{\mathbf{s}}+C_9\bar{\mathbf{p}} \qquad (148)$$

$$\bar{\mathbf{b}}=\frac{i\omega}{c}\,{}_0P(nD_8\,{}_0P+C_8+C_9)\bar{\mathbf{u}}\times\bar{\mathbf{s}}+C_8\bar{\mathbf{m}}+C_{10}\,{}_0P^2(\bar{\mathbf{m}}\bar{\mathbf{s}})\bar{\mathbf{s}} \qquad (149)$$

$$\bar{\mathbf{b}}=n\bar{\mathbf{s}}\times\bar{\mathbf{e}} \qquad (150)$$

$$n\bar{\mathbf{b}}\times\bar{\mathbf{s}}-\bar{\mathbf{e}}=\bar{\mathbf{p}}+n\bar{\mathbf{m}}\times\bar{\mathbf{s}}-i\frac{\omega}{c}\,{}_0Pn[\bar{\mathbf{u}}-(\bar{\mathbf{u}}\bar{\mathbf{s}})\bar{\mathbf{s}}]. \qquad (151)$$

Elimination of $\bar{\mathbf{e}}$ and $\bar{\mathbf{b}}$ yields a system of three equations in the three unknowns $\bar{\mathbf{u}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{m}}$.

Taking the scalar and vector product with $\bar{\mathbf{s}}$, this system becomes:

$$\omega\left[\{C_1+C_2+(2D_4+D_7+2D_3)\,{}_0P^2+D_{11}\,{}_0P^4\}\frac{n^2}{c^2}-{}_0\rho\right]\bar{\mathbf{u}}\bar{\mathbf{s}}$$

$$+i\frac{n}{c}(D_5+D_6\,{}_0P^2+2D_7-C_9)\,{}_0P\bar{\mathbf{p}}\bar{\mathbf{s}}=0 \qquad (152)$$

$$[C_8+C_{10}\,{}_0P^2]\bar{\mathbf{m}}\bar{\mathbf{s}}=0 \qquad (153)$$

$$i\omega\frac{n}{c}\,{}_0P(D_5+D_6\,{}_0P^2+2D_7-C_9)\bar{\mathbf{u}}\bar{\mathbf{s}}-(D_{12}\,{}_0P^2+C_9+1)\bar{\mathbf{p}}\bar{\mathbf{s}}=0. \qquad (154)$$

Hence $\bar{\mathbf{m}}\bar{\mathbf{s}}=0$ and

$$\frac{1}{n^2}=\frac{C_1+C_2}{{}_0\rho c^2}\left[1+\left\{2D_4+D_7+2D_3+D_{11}\,{}_0P^2-\frac{(D_5+D_6\,{}_0P^2+2D_7-C_9)^2}{D_{11}\,{}_0P^2+C_9+1}\right\}\frac{{}_0P^2}{C_1+C_2}\right] \qquad (155)$$

or $\bar{\mathbf{u}}\bar{\mathbf{s}}=\bar{\mathbf{p}}\bar{\mathbf{s}}=0$.

$$\omega^2\left[\{C_2+{}_0P^2(D_4+D_7)\}\frac{n^2}{c^2}+D_8\,{}_0P^3\frac{n}{c^2}-{}_0\rho\right]\bar{\mathbf{u}}\times\bar{\mathbf{s}}$$

$$+i\omega\frac{n}{c}\,{}_0PD_7\bar{\mathbf{p}}\times\bar{\mathbf{s}}-\frac{i\omega}{c}\,{}_0P({}_0PD_8n+C_9)\bar{\mathbf{m}}=0 \qquad (156)$$

$$\frac{i\omega}{c}{}_0P(nD_8\,{}_0P+C_8+C_9-D_7n^2)\bar{\mathbf{u}}\times\bar{\mathbf{s}}+nC_9\bar{\mathbf{p}}\times\bar{\mathbf{s}}+C_8\bar{\mathbf{m}} = 0 \tag{157}$$

$$\frac{i\omega n}{c}[-nD_8\,{}_0P-C_8-C_9+D_7+1]\,{}_0P\bar{\mathbf{u}}\times\bar{\mathbf{s}}-(C_9+1)\bar{\mathbf{p}}\times\bar{\mathbf{s}}-n(C_8-1)\bar{\mathbf{m}} = 0. \tag{158}$$

The characteristic equation of this homogeneous system is of the fourth degree in $\beta = 1/n$, whose coefficients depend on the small parameters λ, μ, ν. At the previous approximation, the equation splits into a pair of quadratic equations,

$$\beta^2-\frac{2D_8\,{}_0P(C_9^2+C_8D_7)}{C_8C_9(C_8-1)}\lambda\beta-\frac{C_9(C_8-1)}{C_8(1+C_9)} = 0 \tag{159}$$

and

$$\beta^2+\frac{2D_8\,{}_0P(D_7+C_9^2)}{C_9(C_8-1)}\lambda\beta-\mu+\left[\frac{D_7^2}{C_9}-D_4-D_7+\frac{D_8^2\,{}_0P^2}{C_8-1}\right]\lambda = 0. \tag{160}$$

Hence,

$$\frac{1}{n^2}\simeq\frac{C_9}{1+C_9}\cdot\frac{C_8-1}{C_8} \tag{161}$$

and

$$\frac{1}{n} = \pm\left(\frac{C_2}{{}_0\rho c^2}\right)^{\frac{1}{2}}\left[1+\left(D_4+D_7-\frac{D_7^2}{C_9}-\frac{D_8^2\,{}_0P^2}{C_8-1}\right)\frac{{}_0P^2}{C_2}\right]^{\frac{1}{2}}-\frac{D_8(D_7+C_9^2)\,{}_0P^3}{C_9(C_8-1)\,{}_0\rho c^2}. \tag{162}$$

(161) gives the velocity of a transverse electromagnetic wave, with arbitrary polarization.

The velocity of the transverse sound wave (162) depends on the sense of propagation (parallel or anti-parallel with ${}_0\mathbf{E}$), since the last term may not be neglected compared with the first one. This effect vanishes for nonmagnetizable materials, for which the velocity in both directions reduces to

$$\frac{1}{n^2} = \frac{C_2}{{}_0\rho c^2}\left[1+\frac{(D_4+D_7)\,{}_0P^2}{C_2}-\frac{D_7^2\,{}_0P^2}{C_2C_9}\right]. \tag{163}$$

The model does not exhibit any dielectric analogous to the Faraday rotation.

6. THE CORRELATION BETWEEN EXPERIMENTAL DATA, MICROSCOPIC MODELS AND CONTINUUM THEORY

It is worthwhile to compare the results provided by the previous theory with experimental data and with conclusions drawn about the same phenomena by other theories at the microscopic level.

The main conclusions of previous sections are:

1. Magnetization seems to appear, at least in this continuum model, as a necessary condition for the medium to exhibit the Cotton–Mouton–Voigt effect in a transverse magnetic field.

2. In a transverse magnetic or electric field, the medium exhibits also a birefringence of the transverse sound waves, while the longitudinal sound wave has a disturbed velocity.
3. The addition of magnetization alone, to an already polarizable isotropic dielectric model, is insufficient for explaining either the Faraday or the Pockels effects.
4. The results relative to the electrooptical Kerr effect, differ only in a quantitative manner from those which could be obtained from Toupin's nonmagnetizable model [1].

It will be shown in this section, that the results obtained from the continuum theory including magnetization, are noncontradictory neither with experimental facts nor with microscopic and quantum theories.

On the contrary, the experimental facts and the other theoretical derivations show that the Voigt–Cotton–Mouton effect and the related acoustical birefringence and velocity changes, depend on the magnetizability of the matter.

(a) *Comparison with experimental data*

Magnetooptical and magnetoacoustical effects. The close relationship between magnetization, magnetic susceptibilities and Cotton–Mouton–Voigt effect were extensively investigated in the period 1930–1940. Most available data are reported by Beams [7] and Schütz [8]. The experimental data show that most dielectrics are weakly magnetizable, and that the value of the classical Cotton–Mouton constant is very small (of order of $10^{-12}\ g^{-1}\ sec^2$). This is in agreement with the theoretical results which show that if C_8 is very large (weakly magnetizable dielectric), the difference between the two velocities (107) (111) of electromagnetic waves in a transverse magnetic field is small.

In special dielectrics, as colloidal suspensions of ferrites in a dielectric jelly, whose magnetizability is much larger, a Cotton–Mouton constant a million times larger than usual has been measured [12]. The sign of the Cotton–Mouton constant cannot be predicted by the theory. In fact, materials have been found with positive Cotton–Mouton constants (aromatic compounds, esters of fatty acids) or negative ones (ethers, saturated alcohols).

Finally, it is an experimental fact [10] that, though the Cotton–Mouton effect may increase in a spectral region where the medium becomes dissipative, it exists in the transparent regions where the dissipation and the chromatic dispersion of the refractive index are zero or negligible.

Unpublished data of Dobbs, Chick and Fitzgerald [19], show an increase of the velocity of the longitudinal sound wave of 0·53 per cent (at 300 K, in a transverse magnetic field of 3 kilogauss), in a Nickel sample. Though this figure is very small, it is significant since the measurements of the change in the velocity, due to the magnetization, are more accurate than the velocity measurements themselves. It is relevant for our continuum theory that the effect could only be measured in a highly magnetizable medium. Data published by Goodrich and Lange [24] lead to similar conclusions. Acoustical birefringences, induced by a transverse magnetic field, were also observed by Watkins and Feher [22], Wang and Crow [23].

In cases where metallic samples are used, the interpretation and comparison of the experimental data with our theoretical results is more tedious, because of the conductivity of the medium. An attempt was made by Lange [14] to separate both effects, namely the influence of the magnetic field on the elastic behaviour, and on the charge-carrier electrons responsible for the conduction.

In metals, the latter effect is often dominant, but the former exists (and becomes dominant in insulators and semiconductors), which is in agreement with previous theoretical conclusions drawn about dielectrics.

Electrooptical and electroacoustical effects. Exhaustive investigations and experimental data about the transverse electrooptical Kerr effect can be found in [7]. These data are in good agreement with the Langevin–Born–Van Vleck theory, which derives the value of the Kerr constant from the dielectric polarizability of single molecule. Hence, the Kerr effect is obviously related to the *polarization* of the medium, in a way similar to the relationship between the Cotton–Mouton effect and the *magnetization*. No significant experimental data seem available about the influence of the magnetizability neither on the Kerr effect, nor on the velocities of the acoustical waves in an electric field. This is not surprising, since careful estimates of Van Vleck [16] have shown that the possible influence of an electric field on the magnetizability of matter is very weak (an electric field of 10,000 V would alter the magnetic susceptibility of a typical molecule, at ordinary temperatures, by only about one part in 10^8, which is much smaller than the corresponding effect of a magnetic field responsible for the Cotton–Mouton and related effects.)

No relevant data have been found related to the effect of an electric field upon the velocity of sound waves or to the effect described by formula (162). Therefore it would be advisable to perform some experiments in strong longitudinal electric field, within a highly magnetizable medium such as colloidal suspension of ferrite in a dielectric jelly, especially in order to check (162).

(b) *Comparison with other theories*

Classical and semiquantum theories of the magneto- and electrooptical effects at the molecular level including connection with bulk magnetization and polarization, have been given earlier [7, 9, 16, 26 and 13]. In particular, the relationship between microscopic measurements based on the Cotton–Mouton effect and the magnetizability of a molecular unit, and on the other hand between the Kerr effect and the polarizability of the unit was understood and made computation of these molecular parameters possible from the experimental data [7, 11 and 13].

A classical wave-mechanical approach was presented by Tilleu [17]. Out of the quantum averaging for polarization and magnetization, it can be seen that all relevant contributions for the Cotton–Mouton–Voigt effect vanish with the magnetization and the static magnetic susceptibility of the medium.

A quantum field description of the magnetooptical birefringence due to Atkins [18] shows again that the magnetooptical transverse birefringence vanishes with the static magnetic susceptibility, the permanent magnetic moment of the molecule (which also contributes to the magnetization of the matter) and higher-order polarization-magnetization coupling terms.

Parallel conclusions can be drawn out of the treatment of the Kerr effect by the same author [15].

7. CONCLUSIONS

Using expressions derived by Mayne and Boulanger for the momentum and energy balance in a polarizable but also *magnetizable* continuum, we have investigated the photoelastic, magnetooptic and electrooptic response of the model. The theory provides adequate results for the transverse birefringence effects, of electromagnetic (light waves) and mechanical (sound waves) nature. Good support for these results is found in experimental data, and in quantum and quantum field models.

The theory does not include either the Faraday or the Pockels rotation. This lack is not in disagreement with experiment, since the materials which show these effects are always either dissipative or anisotropic. Addition of dependence in the constitutive relations of time-derivatives of polarization and/or magnetization, proposed by Boulanger [28], provides a new model which exhibits, in a longitudinal magnetic field, optical and mechanical circular birefringence (Faraday effect) and circular dichroism (namely a difference between the absorptions of right and left polarized waves).

These results seem to be supported by experimental evidence. It has indeed been recognized [26, 27] that the existence of Faraday rotation is related to Zeeman splitting, which in turn implies not only circular birefringence, but also circular dichroism [9, 10, 29, 15 and 30]. The quantum investigation in a ferromagnetic medium [20] also indicates strong correlation between the *Faraday effect*, *absorption*, and *magnetization*. The analysis is complicated by the ohmic loss due to conductivity [31, 32]. Finally, Boulanger's model predicts also a Faraday effect with dichroism for the transverse sound waves. The existence of this effect has been shown and explained by Aminov [21] and also by Aubauer [33] for a magnetized, conductive medium.

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Абстракт—С динамической точки зрения, исходя из уравнений равновесия и конститутивных зависимостей, исследуются магнитооптические, электрооптические и фотоуругие поведелия упругой, поляризуемой и поддающейся намагнитованию, изотропной сплошной среды. Наиболее оригинальным результатом теории является факт, что упругая среда проявляет эффект Комтона-Мутона, наряду с линейным двойным лучепреломлением поперечных звуковых волн. Сравнивается это явление с экспериментальными данными и результатами квантовой теории. Как должно было ожидать, сплошная среда не проявляет вращения фарадея.